Section 16.3 The Fundamental theorem for Line Integrals

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Fundamental theorem of Calculus

In Calculus 1, we have the Fundamental theorem of Calculus that told us how to evaluate definite integrals.

Fundamental theorem of Calculus

$$\int_a^b f'(x)dx = f(x)|_a^b = f(b) - f(a)$$

The Fundamental theorem for Line Integrals

Theorem

Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \leq t \leq b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Note:

$\vec{r}(a)$ represent the initial point on C and $\vec{r}(b)$ represent the end point on C

This theorem will hold regardless of the number of variables in the function.

Integration is path independent.

$f(x,y)$ $f(x,y,z)$ $f(x_1, x_2 \cdots x_n)$

∇f $f(\ \ )$

Proof of the theorem

Prove: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

We work in 3-D for this proof, but it can be done in any dimensions

Let's start by computing the line integral

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Here $\vec{r}(t) = < x(t), y(t), z(t) >$ and $\vec{r}'(t) = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >$

$\nabla f(\vec{r}(t)) = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$

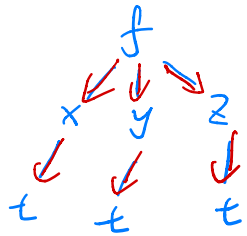
So $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$

Use chain rule to simplify the integrand

$= \int_a^b \frac{d}{dt}(f(\vec{r}))dt = \int_a^b f' dt = f(\vec{r}(b)) - f(\vec{r}(a))$

By the Fundamental Theorem of Calculus

$= f(\vec{r}(b)) - f(\vec{r}(a))$



Example 1

$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\text{end point})) - f(\vec{r}(\text{initial point}))$

Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1, 0.5, 2)$ and ends at $(2, 1, -1)$

Solution:

let $\vec{r}(t)$ $a \le t \le b$ be any path that start at $(1, 0.5, 2)$ and end at $(2, 1, -1)$ then

$\vec{r}(a) = \langle 1, 0.5, 2 \rangle$ and $\vec{r}(b) = \langle 2, 1, -1 \rangle$

The integral $\int_C \nabla f \cdot d\vec{r} = \underbrace{f(2, 1, -1)}_{f(\vec{r}(b))} - \underbrace{f(1, 0.5, 2)}_{f(\vec{r}(a))}$

point (a, b)
$\vec{r} = \langle a, b \rangle$
$f(\vec{r}) = f(a, b)$

$= \cos(2\pi) + \sin(\pi) - 2(1)(-1) - \left(\cos(\pi) + \sin\left(\frac{\pi}{2}\right) - 1\left(\frac{1}{2}\right)\right)$

$= 1 + 0 + 2 - (-1 + 1 - 1)$

$= 4$

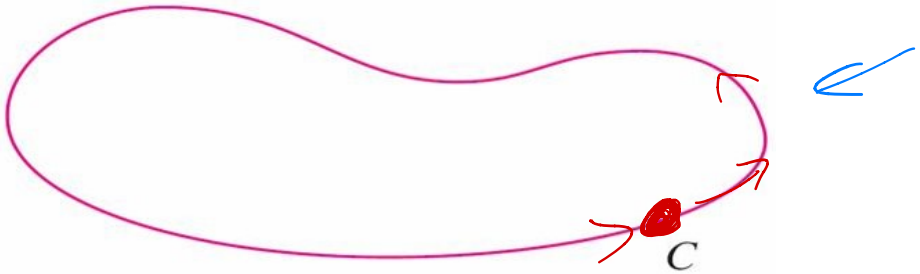
Definitions

Definition

Suppose $\vec{F}$ is a continuous vector field in some domain D. $\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** if $\int_{C1} \vec{F} \cdot d\vec{r} = \int_{C2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D with the same initial and end points.

Definition

A path C is called **closed** if its initial and end points are the same point.
e.g. a circle is a closed path.



$\int_a^a \nabla f \cdot d\vec{r} = ? \quad f(\vec{r}(a)) - f(\vec{r}(a))$

$= 0$.

Theorem

Theorem

If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

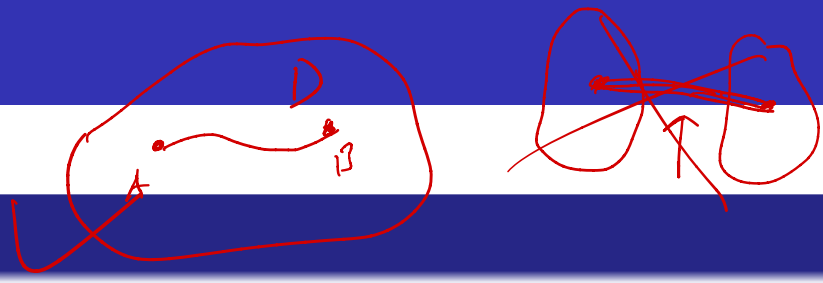
Proof:

if $\vec{F}$ is conservative, then it has a potential function f such that $\vec{F} = \nabla f$

So $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$

Since $\int_C \nabla f \cdot d\vec{r}$ is independent of path, so is $\int_C \vec{F} \cdot d\vec{r}$

Theorem



Definition

A region D is **open** if it doesn't contain any of its boundary points. A region is **connected** if we can connect any two points in the region with a path that lies completely in D

Theorem

If $\vec{F}$ is continuous vector field on an open connected region D and if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path (for any path in D). Then $\vec{F}$ is a conservative vector field on D. i.e. there exists a function f such that $\vec{F} = \nabla f$

Question

Questions:

1. Given a vector field $\vec{F}$, is there any way of determining if it is a conservative vector field?

For example, $\vec{F} =< P(x,y), Q(x,y) >$, how to determine if there is a function f with $\nabla f =< f_x, f_y >$, such that $f_x = P(x,y)$ and $f_y = Q(x,y)$?

2. If we know that $\vec{F}$ is a conservative vector field. How do we go about finding a potential function f for the vector field?

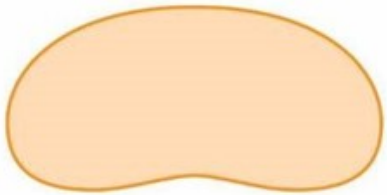
Definition

Definition

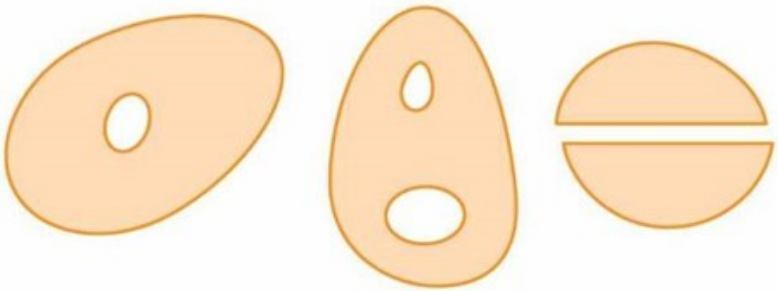
A path C is **Simple** if it doesn't cross itself

Definition

A region D is **simply-connected** if it is connected and it contains no holes



simply-connected region



regions that are not simply-connected

Theorem

Here is how to identify if a given vector field $\vec{F}$ is conservative:

Theorem

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open and simply-connected region D. If P and Q have continuous first order partial derivatives in D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the vector $\vec{F}$ is conservative.

The reason behind this is. If $\vec{F}$ is conservative, then

$\vec{F} = \langle P, Q \rangle = \nabla f = \langle f_x, f_y \rangle$

So $P = f_x$, $Q = f_y$ if $P_y = f_{xy}$, $Q_x = f_{yx}$

So $P_y = f_{xy} = f_{yx} = Q_x$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Example 2

Determine if the following vector fields are conservative or not

(a). $\vec{F}(x,y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$

P $\quad$ Q

$\frac{\partial P}{\partial y} = -x$

$\frac{\partial Q}{\partial x} = -y$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ then $\vec{F}$ is not conservative.

(b). $\vec{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$

P $\quad$ Q

$\frac{\partial P}{\partial y} = 2xe^{xy}x + x^2e^{xy} + x^2ye^{xy}x = 3x^2e^{xy} + x^3ye^{xy}$

$\frac{\partial Q}{\partial x} = 3x^2e^{xy} + x^3e^{xy}y$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so $\vec{F}$ is conservative.

Find the potential function f

Now that we know how to identify if a given vector field is conservative.
Next, we need to address how to find a potential function f for the vector field.

Assume the vector field is conservative and a potential function $f(x, y)$ exist. we have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j} = \vec{F}$$

i.e. $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$

Integrating: $f = \int P(x, y)dx$ or $f = \int Q(x, y)dy$

$$f = \int \frac{\partial f}{\partial x}\,dx \quad \text{or} \quad f = \int \frac{\partial f}{\partial y}\,dy$$

Example 3

Determine if the vector field $\vec{F} = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$ is conservative and find a potential function for the vector field if it is conservative

Solution: $P = 2x^3y^4 + x$ $\quad \frac{\partial P}{\partial y} = 8x^3y^3$

$Q = 2x^4y^3 + y$ $\quad \frac{\partial Q}{\partial x} = 8x^3y^3$

So $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so $\vec{F}$ is conservative.

then $\vec{F} = \nabla f$ $\quad \langle P, Q \rangle = \langle f_x, f_y \rangle$

$P = f_x$, $Q = f_y$

$$f(x,y) = \int P\,dx = \int (2x^3y^4 + x)\,dx = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y)$$

$$f(x,y) = \int Q\,dy = \int (2x^4y^3 + y)\,dy = \frac{1}{2}x^4y^4 + \frac{1}{2}y^2 + g(x)$$

$$\Rightarrow \boxed{f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C}$$