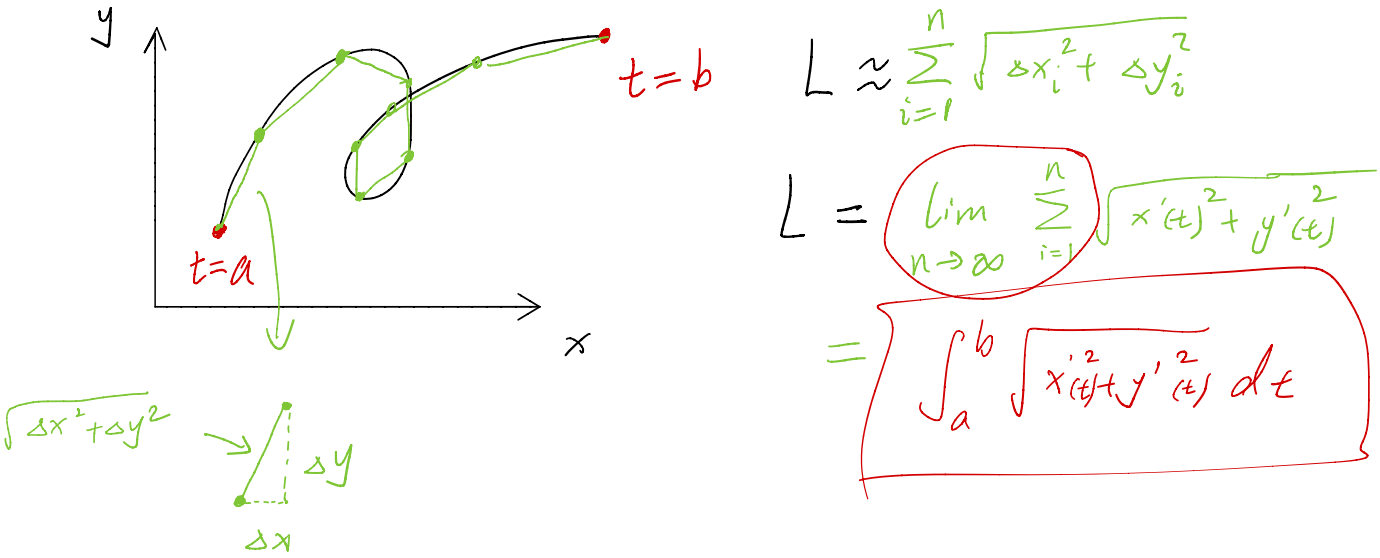
1. 13.3 Arc Length

recall the formula for arc length of a parametrically defined curve in a plane discussed in section 10.2: $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$ Arc length in 3-Space can be found similarly.



Theorem 1.1. *The* **arc length** *of a curve, C, defined by the vector function $\vec{r} = \langle x(t), y(t), z(t)\rangle$ from $t = a$ to $t = b$ is*

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt.$$

$$= |\langle x'(t), y'(t), z'(t)\rangle| = |\vec{r}\,'(t)|$$

Definition 1.1. *The* **arc length function** *with initial value $t = a$ is*

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\, du = \int_a^t |\vec{r}\,'(u)|\, du.$$

Remark 1.1. *Notice that $\frac{ds}{dt} = |\vec{r}\,'(t)|$, so $ds = |\vec{r}\,'(t)|\, dt$.*

$$\frac{d}{dt} s(t) = \frac{d}{dt}\int_a^t |r'(u)|\, du$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

$$ds = |\vec{r}\,'(t)|\, dt$$

Remark 1.2. *If we can rewrite $s = s(t)$ so that $t = t(s)$ we use this* **parametrize a curve with respect to arc length:** $\vec{r}(t(s))$.

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

Example 1.1. *Let* $\vec{r}(t) = e^t \sin t\, \vec{i} + e^t \cos t\, \vec{j} + \sqrt{2}e^t \vec{k}$.

(1) Find the length of the curve for $0 \le t \le 3$.

$$\vec{r}\,'(t) = \langle e^t \sin t + \cos t\, e^t,\ e^t \cos t - \sin t\, e^t,\ \sqrt{2}e^t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + (\sqrt{2}e^t)^2}$$

$$= \sqrt{[e^t(\sin t + \cos t)]^2 + [e^t(\cos t - \sin t)]^2 + 2(e^t)^2}$$

$$= \sqrt{e^{2t}(\sin^2 t + 2\sin t \cos t + \cos^2 t) + e^{2t}(\cos^2 t - 2\sin t \cos t + \sin^2 t) + 2e^{2t}}$$

$$= e^t \sqrt{4} = 2e^t$$

$$L = \int_0^3 2e^t\, dt = 2\int_0^3 e^t\, dt = 2\left(e^t\big|_0^3\right) = 2(e^3 - 1) = \boxed{2e^3 - 2}$$

(2) Find arc length function of the curve measured from $P(0, 1, \sqrt{2})$ *and in the direction of increasing* t.

$$\vec{r}(t) = e^t \sin t\, \vec{i} + e^t \cos t\, \vec{j} + \sqrt{2}e^t \vec{k}$$

$t \stackrel{?}{=} 0$ to stay at $(0, 1, \sqrt{2})$

$$L = \int_0^t 2e^u\, du = 2\int_0^t e^u\, du = 2\left(e^u\big|_0^t\right) = 2(e^t - e^0)$$

$$= \boxed{2e^t - 2}$$

(3) Parametrize the curve with respect to arc length.

$\vec{r}(t)$ $\vec{r}(s)$

$$s = 2e^t - 2$$

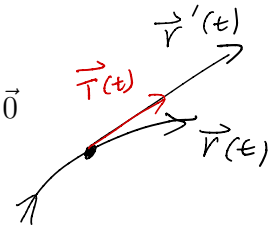
$$\frac{s+2}{2} = e^t$$

$$\ln\left(\frac{s+2}{2}\right) = t$$

$$\vec{r}(t) = e^t \sin t\, \vec{i} + e^t \cos t\, \vec{j} + \sqrt{2}e^t \vec{k}$$

$$\vec{r}(s) = \left(\frac{s+2}{2}\right)\sin\left(\ln\frac{s+2}{2}\right)\vec{i} + \left(\frac{s+2}{2}\right)\cos\left(\ln\frac{s+2}{2}\right)\vec{j} + \sqrt{2}\left(\frac{s+2}{2}\right)\vec{k}$$

2. Curvature



Definitions 2.1.

(1) *A parametrization $\vec{r}$ is* **smooth** *on an interval if $\vec{r}\,'$ is continuous and $\vec{r}\,' \neq \vec{0}$ on the interval.*

(2) *Recall the* **unit tangent vector** *to a curve is* $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

(3) *The* **curvature** *of a curve is* $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$, *the rate of change of the direction of the unit tangent vector with respect to the arc length.*

Theorem 2.1. *Alternate formulas for curvature are* $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$

(1) $\kappa = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$ $\quad \kappa = \left|\dfrac{d\vec{T}}{ds}\right| = \left|\dfrac{d\vec{T}}{dt}\dfrac{dt}{ds}\right| = \left|\dfrac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$

(2) $\kappa = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

(3) *For a plane curve with* $y = f(x)$, $\kappa = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

$$\kappa = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

Example 2.1. *Find the curvature of* $\vec{r}(t) = 3t\vec{i} + 4\sin t\,\vec{j} + 4\cos t\,\vec{k}$

$\vec{r}\,'(t) = \langle 3, 4\cos t, -4\sin t \rangle$

$|\vec{r}\,'(t)| = \sqrt{3^2 + (4\cos t)^2 + (-4\sin t)^2} = \sqrt{9 + 16\cos^2 t + 16\sin^2 t}$

$= \sqrt{9 + 16(\cos^2 t + \sin^2 t)} = \sqrt{25} = 5$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 3, 4\cos t, -4\sin t \rangle}{5}$

$= \left\langle \frac{3}{5}, \frac{4}{5}\cos t, -\frac{4}{5}\sin t \right\rangle$

$\vec{T}\,'(t) = \left\langle 0, -\frac{4}{5}\sin t, -\frac{4}{5}\cos t \right\rangle$

$|\vec{T}\,'(t)| = \sqrt{0^2 + \frac{16}{25}\sin^2 t + \frac{16}{25}\cos^2 t} = \sqrt{\frac{16}{25}(\sin^2 t + \cos^2 t)} = \frac{4}{5}$

$\kappa = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \dfrac{\frac{4}{5}}{5} = \dfrac{4}{25}$

$\kappa = \dfrac{\sqrt{2}(e^t - e^{-t})}{(e^t - e^{-t})^3} = \dfrac{\sqrt{2}}{(e^t - e^{-t})^2}$

Example 2.2. *Use the forumula* $\kappa = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ *to find the curvature of* $\vec{r}(t) = \sqrt{2}t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$

$\vec{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$

$\vec{r}''(t) = \langle 0, e^t, e^{-t} \rangle$

$(-e^{-t})' = (-e^{-t})(-1) = e^{-t}$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix}$$

$= [(e^t)(e^{-t}) - (e^t)(-e^{-t})]\vec{i} - (\sqrt{2}e^{-t} - (0)(-e^{-t}))\vec{j} + (\sqrt{2}e^t - (0)(e^t))\vec{k}$

$= (e^{t-t} + e^{t-t})\vec{i} - \sqrt{2}e^{-t}\vec{j} + \sqrt{2}e^t\vec{k}$

$= 2\vec{i} - \sqrt{2}e^{-t}\vec{j} + \sqrt{2}e^t\vec{k}$

$\star \quad |\vec{r}' \times \vec{r}''| = \sqrt{2^2 + (\sqrt{2}e^{-t})^2 + (\sqrt{2}e^t)^2} = \sqrt{4 + 2e^{-2t} + 2e^{2t}}$

$= \sqrt{2(2 + e^{-2t} + e^{2t})} = \sqrt{2(e^{2t} + 2 + e^{-2t})}$

$= \sqrt{2}\sqrt{(e^t)^2 + 2(e^t)(e^{-t}) + (e^{-t})^2} = \sqrt{2}\sqrt{(e^t + e^{-t})^2} = \sqrt{2}(e^t + e^{-t})$

$|\vec{r}'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + (e^t)^2 + (e^{-t})^2} = \sqrt{(e^t + e^{-t})^2}$

$= e^t + e^{-t}$

Example 2.3. *Use the forumula* $\kappa = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$ *to find the curvature of* $y = x^3$ *at point* $(1, 1)$

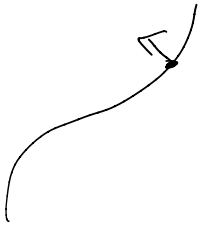
$y = x^3 \qquad f(x) = x^3$

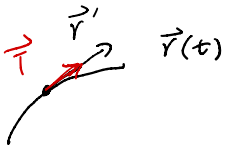
$f'(x) = 3x^2 \qquad f''(x) = 6x$

at $(1,1)$ $\quad x = 1$

$f'(1) = 3 \qquad f''(1) = 6$

$$\kappa = \frac{|6|}{[1 + 3^2]^{\frac{3}{2}}} = \frac{6}{10^{\frac{3}{2}}}$$





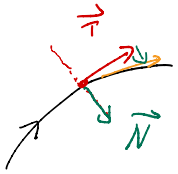
if $|\vec{r}(t)| = C \Leftrightarrow \vec{r}'$ is orthogonal to $\vec{r}$

$|\vec{T}(t)| = 1 \Leftrightarrow \vec{T}'$ is orthogonal to $\vec{T}$

3. Normal and Binormal Vectors

Definitions 3.1. *We define the following at a point P on the curve C where $\kappa \neq 0$.*

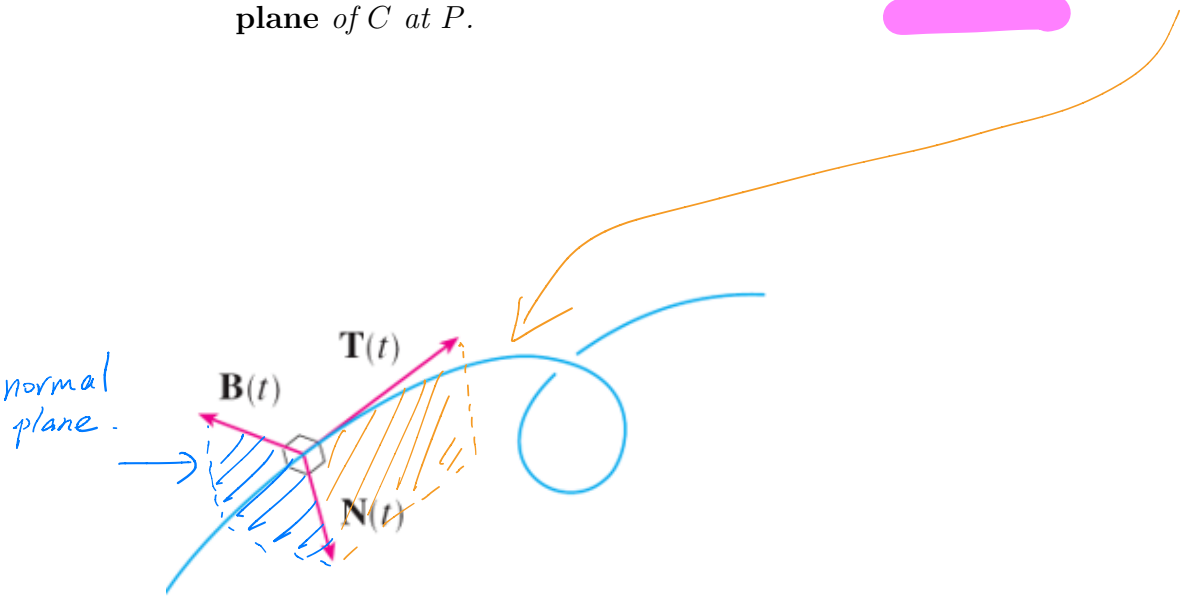
(1) *The* **(principal) unit normal vector**, $\vec{N}(t)$, *is* $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$.



(2) *The* **binormal vector**, $\vec{B}(t)$, *is* $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

(3) *The plane through the point P on C determined by $\vec{N}$ and $\vec{B}$ is the* **normal plane** *of C at P.*

(4) *The plane through the point P on C determined by $\vec{T}$ and $\vec{N}$ is the* **osculating plane** *of C at P.*



Example 3.1. *Find the vectors* $\vec{T}, \vec{N}, \vec{B}$ *for* $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$

$$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \frac{\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle}{\sqrt{\left[\left(\frac{1}{\sqrt{2}}\right)(-\cos t)\right]^2 + \left[\left(\frac{1}{\sqrt{2}}\right)(-\sin t)\right]^2 + 0^2}} = \frac{\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle}{\frac{1}{\sqrt{2}}}$$

$$= \langle -\cos t, -\sin t, 0 \rangle$$

$$\vec{B} = \vec{T} \times \vec{N}$$

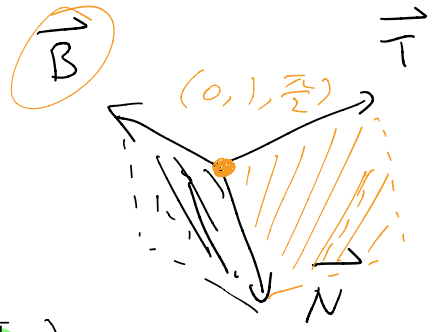
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}}(-\sin t) & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left\langle \frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}} \right\rangle$$

plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

point (x_0, y_0, z_0)

$\vec{n} = \langle a, b, c \rangle$

Example 3.2. *Find the equation for the normal plane and the osculating plane to* $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ *at the point* $(0, 1, \frac{\pi}{2})$



point $(0, 1, \frac{\pi}{2})$

at this point, what is $t = \frac{\pi}{2}$

For the normal plane, the normal vector $\vec{n} = \vec{T}$

$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$

at $t = \frac{\pi}{2}$ $\quad \vec{T}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$

we could use $\vec{n} = \langle -1, 0, 1 \rangle$

normal plane:

$$-1(x-0) + 0(y-1) + (z - \tfrac{\pi}{2}) = 0$$

$$-x + z - \tfrac{\pi}{2} = 0$$

$$\boxed{z = x + \tfrac{\pi}{2}}$$

$\vec{B}(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$

at $t = \frac{\pi}{2}$ $\quad \vec{B}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$

we can use $\vec{n} = \langle 1, 0, 1 \rangle$ $\quad$ point $P(0, 1, \frac{\pi}{2})$

osculating plane

$$(x-0) + 0(y-1) + (z - \tfrac{\pi}{2}) = 0$$

$$\boxed{z = -x + \tfrac{\pi}{2}}$$