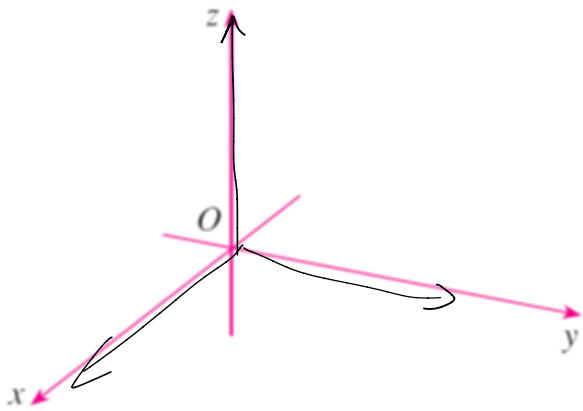
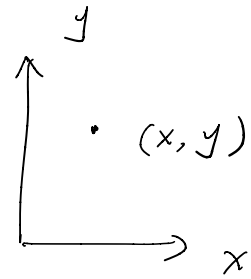
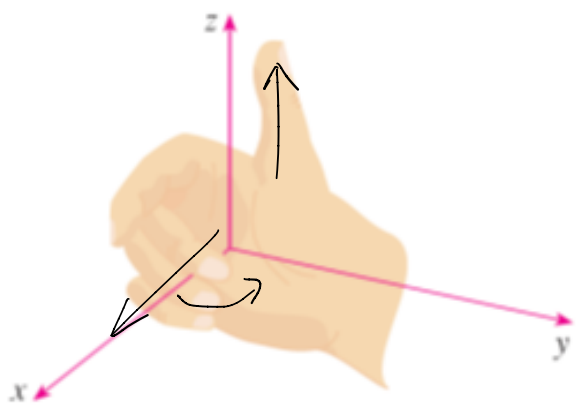
1. 12.1 3D Coordinate Systems

(1) 3D Coordinate Axis



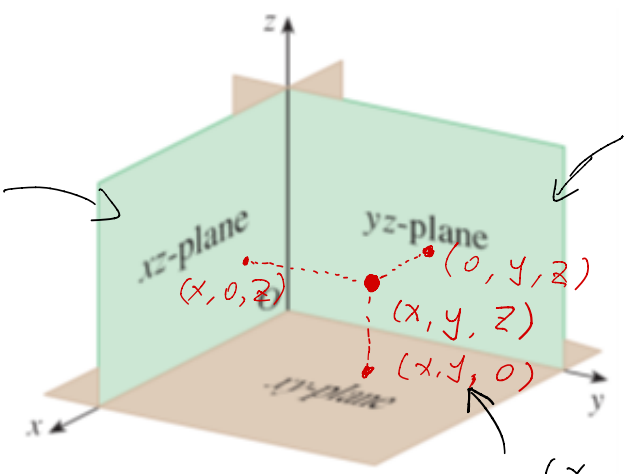


(2) Right-hand rule The direction of the z-axis is determined by the right-hand rule



Point our fingers at the direction of x-axis, curl our fingers towards the y-axis, then the thumb will point to the z-axis

(3) Coordinate Planes



(x, 0, z)

xz-plane

$\boxed{y = 0}$

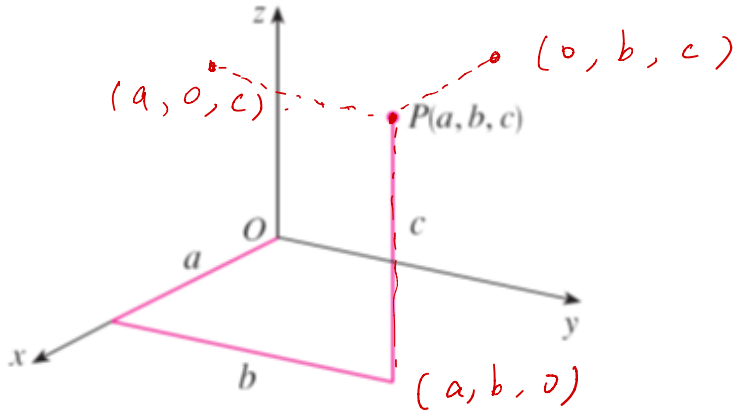
(0, y, z)

yz-plane $\boxed{x = 0}$

(x, y, 0)

xy-plane $z = 0$

(4) Projection of the point $P(a, b, c)$ on to the plane $z = 0$ is $(a, b, 0)$. Similarly there are projections of a point on to any coordinate plane.

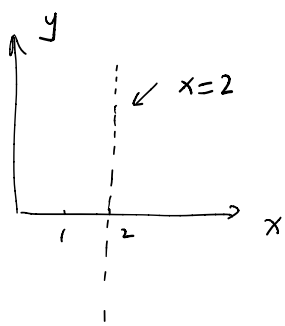


2. Surface

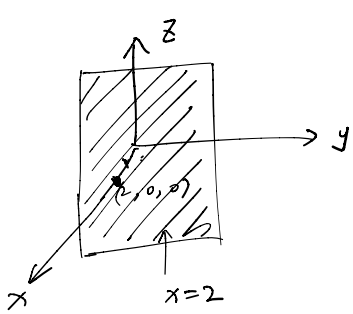
The graph of an equation involving x, y, and z is a **surface** in $\mathbb{R}^3$.

Remark 2.1.
(1) $x = 0$ is the yz-plane.
(2) $y = 0$ is the xz-plane.
(3) $z = 0$ is the xy-plane.

Example 2.1. *What does $x = 2$ represent in the plain ($\mathbf{R}^2$)?*



Example 2.2. *What does $x = 2$ represent in the space ($\mathbf{R}^3$)?*



Distance in 2D $P_1(x_1, y_1)$ $P_2(x_2, y_2)$

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

3. Distance

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 3.1. *Find the distance from the point $P(2, -1, 7)$ to $Q(1, 0, 2)$*

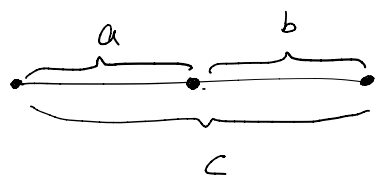
$$|PQ| = \sqrt{(1-2)^2 + (0-(-1))^2 + (2-7)^2}$$
$$= \sqrt{1 + 1 + 25}$$
$$= \sqrt{27}$$
$$= 3\sqrt{3}$$

Example 3.2. *Are the points $D(0, -3, 3)$, $E(1, 1, 2)$, $F(3, 5, 1)$ on a straight line? If not, do they create a right triangle?*

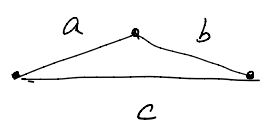
$$|DE| = \sqrt{(1-0)^2 + (1-(-3))^2 + (2-3)^2} = \sqrt{18}$$

$$|DF| = \sqrt{(3-0)^2 + (5-(-3))^2 + (1-3)^2} = \sqrt{77}$$

$$|EF| = \sqrt{(3-1)^2 + (5-1)^2 + (1-2)^2} = \sqrt{21}$$



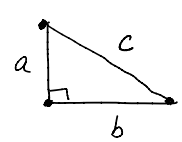
$a + b = c$



$a + b > c$

$\sqrt{18} + \sqrt{21} < \sqrt{77}$ not on a straight line.

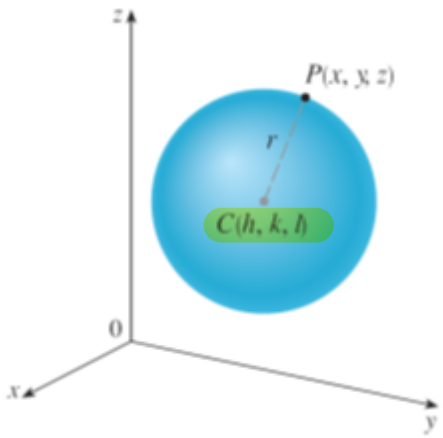
$(\sqrt{18})^2 + (\sqrt{21})^2 \neq (\sqrt{77})^2$

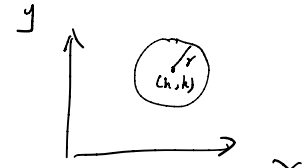


$a^2 + b^2 = c^2$

not a right triangle.

4. Sphere





$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

circle

distance from (x, y, z) to (h, k, l) are always r.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Example 4.1. *Find the center and radius of the sphere:* $x^2+y^2+z^2+2x-4y-6z+5 = 0$.

$$x^2+2x+1 + y^2-4y+4 + z^2-6z+9 + 5$$

$$= 1 + 4 + 9$$

$$(x+1)^2 + (y-2)^2 + (z-3)^2 + 5 = 14$$

$$\boxed{x^2 + ax + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2}$$

$$(x+1)^2 + (y-2)^2 + (z-3)^2 = 3^2$$

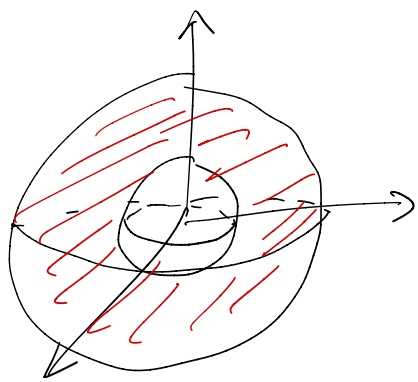
center $(-1, 2, 3)$

radius 3.

Example 4.2. *Use inequalities to describe the region between the spheres with center* $(0, 0, 0)$ *and radii* 2 *and* 5.

sphere of radius 2. $x^2 + y^2 + z^2 = 2^2$

sphere of radius 5 $x^2 + y^2 + z^2 = 5^2$



region $\{(x, y, z) \mid 4 \leq x^2 + y^2 + z^2 \leq 25\}$.