A Review of Probabilistic Data Assimilation Methods for Online State Estimation

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Outline

Data Assimilation

Data assimilation is a mathematical discipline that seeks to optimally combine theory (usually in the form of a numerical model) with observations.

- Objective: determine information about the unknown quantity in numerical model given observations.
- Used in: **state estimation**, determining initial conditions, parameter estimation.
- Two key problems: **filtering**(on-line) and smoothing(off-line)
- Objective of this presentation: Review five filtering methods for state estimation.

Discrete Time Set-up for Filtering

- Initial state x_0 is given.
- Dynamic equation

$$x_{t+1} = f(x_t) + v_t$$

where $t \in \mathbb{N}$, $f \in C(\mathbb{R}^{n_x}, \mathbb{R}^{n_x})$,$\{v_t\}_{t\in\mathbb{N}} \overset{i.i.d}{\sim} N(0, Q)$

- Measurement equation

$$y_{t+1} = h(x_{t+1}) + w_{t+1}$$

where $t \in \mathbb{N}$, $h \in C(\mathbb{R}^{n_x}, \mathbb{R}^{n_y})$, $\{w_t\}_{t\in\mathbb{N}} \overset{i.i.d}{\sim} N(0, R)$

- Goal: Estimate x_{t+1} based on noisy measurement $y_1 = Y_1, \dots, y_{t+1} = Y_{t+1}$

Notation

- Random Variables x_0, x_t, y_t, v_t, w_t for $t \in \mathbb{Z}^+$
- Markov Chain $x_{0:t+1} = \{x_0, x_1, x_2, \ldots, x_{t+1}\}$
- Data $Y_{1:t+1} = \{y_1 = Y_1, \ldots, y_{t+1} = Y_{t+1}\}$
- Transition Density $p(x_{t+1}|x_t)$
- Prior $p(x_{t+1}|y_{1:t})$
- Likelihood $p(y_{t+1}|x_{t+1})$
- Posterior $p(x_{0:t+1}|y_{1:t+1})$
- Filtering Density $p(x_{t+1}|y_{1:t+1})$
- Mean $m_t = E[x_t|y_{1:t} = Y_{1:t}]$
- Predicted Mean $\hat{m}_{t+1} = E[x_{t+1}|y_{1:t} = Y_{1:t}]$
- Covariance $C_t = Cov[x_t|y_{1:t} = Y_{1:t}]$
- Predicted Covariance $\hat{C}_{t+1} = Cov[x_{t+1}|y_{1:t} = Y_{1:t}]$

A Probabilistic View of Dynamic System

- Initial guess $x_0 \sim N(m_0, C_0)$ is given.
- Dynamic equation $x_{t+1} = f(x_t) + v_t, \{v_t\}_{t \in \mathbb{N}} \overset{i.i.d}{\sim} N(0, Q)$ at fixed $x_t = X_t$ describes the transition density of x_{t+1}

$$p(x_{t+1}|x_t = X_t) = N(x_{t+1} - f(X_t); 0, Q)$$

$$\propto exp(-\frac{1}{2}|Q^{-1/2}(x_{t+1} - f(X_t))|^2)$$

- Measurement equation $y_{t+1} = h(x_{t+1}) + w_{t+1}, \{v_t\}_{t \in \mathbb{N}} \overset{i.i.d}{\sim} N(0, R)$ at fixed $x_{t+1} = X_{t+1}$ describes the likelihood of y_{t+1}

$$p(y_{t+1}|x_{t+1} = X_{t+1}) = N(y_{t+1} - h(X_{t+1}); 0, R)$$

$$\propto exp(-\frac{1}{2}|R^{-1/2}(y_{t+1} - h(X_{t+1}))|^2)$$

Filtering Problem from Bayesian Approach

- Objective of filtering: determining $p(x_{t+1}|y_{1:t+1})$
- Compute $p(x_{t+1}|Y_{1:t+1})$ sequentially in time in two steps:
  1. Prediction: $p(x_t|Y_{1:t}) \to p(x_{t+1}|Y_{1:t})$

$$p(x_{t+1}|Y_{1:t}) = \int_{\mathbb{R}^{n_x}} p(x_{t+1}|Y_{1:t}, x_t)p(x_t|Y_{1:t})dx_t$$

$$= \int_{\mathbb{R}^{n_x}} p(x_{t+1}|x_t)p(x_t|Y_{1:t})dx_t$$

  2. Update: $p(x_{t+1}|Y_{1:t}) \to p(x_{t+1}|Y_{1:t+1})$

$$p(x_{t+1}|Y_{1:t+1}) = \frac{p(Y_{t+1}|x_{t+1}, Y_{1:t})p(x_{t+1}|Y_{1:t})}{p(Y_{t+1}|Y_{1:t})}$$

$$\propto p(Y_{t+1}|x_{t+1})p(x_{t+1}|Y_{1:t})$$

Linear Gaussian State Space Model

- Initial guess $x_0 \sim N(m_0, C_0)$ is given.
- Dynamic equation:

$$x_{t+1} = Fx_t + v_t$$

$v_t \overset{i.i.d}{\sim} N(0, Q), F \in R^{n_x \times n_x}$

- Measurement equation:

$$y_{t+1} = Hx_{t+1} + w_{t+1}$$

$w_{t+1} \overset{i.i.d}{\sim} N(0, R), H \in R^{n_y \times n_x}$

Kalman Filter

- Given filtering distribution at time t:

$$p(x_t | Y_{1:t}) = N(x_t; m_t, C_t)$$

- Both Prediction and Update steps preserve Gaussianity.

$$prior : p(x_{t+1} | Y_{1:t}) = N(x_{t+1}; \hat{m}_{t+1}, \hat{C}_{t+1})$$

$$filtering : p(x_{t+1} | Y_{1:t+1}) = N(x_{t+1}; m_{t+1}, C_{t+1})$$

- Derivation objective: Derive the map of mean and covariance in prediction and update steps.

$$m_t, C_t \xrightarrow{\text{prediction}} \hat{m}_{t+1}, \hat{C}_{t+1} \xrightarrow{\text{update}} m_{t+1}, C_{t+1}$$

Kalman Filter: Derivation of Prediction Mapping

Derive the map: $m_t, C_t \xrightarrow{\text{prediction}} \hat{m}_{t+1}, \hat{C}_{t+1}$ from dynamic equation

$$x_{t+1} = Fx_t + v_t, v_t \overset{i.i.d}{\sim} N(0, Q)$$

- Compute the predicted mean,

$$\hat{m}_{t+1} = \mathbb{E}[x_{t+1}|Y_{1:t}] = \mathbb{E}[Fx_t|Y_{1:t}] + \mathbb{E}[v_t|Y_{1:t}] = Fm_t$$

- Compute the predicted covariance.

$$\hat{C}_{t+1} = \mathbb{E}[(x_{t+1} - \hat{m}_{t+1})(x_{t+1} - \hat{m}_{t+1})|Y_{1:t}] =$$

$$\cdots = F\mathbb{E}[(x_t - m_t)(x_t - m_t)|Y_{1:t}]F^T + Q$$

$$= FC_tF^T + Q$$

Kalman Filter: Derivation of Update Mapping

Derive the map: $\hat{m}_{t+1}, \hat{C}_{t+1} \xrightarrow{\text{update}} m_{t+1}, C_{t+1}$

- By Bayes theorem,

$$p(x_{t+1}|y_{1:t+1}) \propto p(y_{t+1}|x_{t+1})p(x_{t+1}|y_{1:t})$$

- Equating the exponential term

$$exp(-\frac{1}{2}|C_{t+1}^{-1/2}(x_{t+1} - m_{t+1})|^2)$$

$$\propto exp(-\frac{1}{2}|R^{-1/2}(y_{t+1} - H(x_{t+1}))|^2)exp(-\frac{1}{2}|\hat{C}_{t+1}^{-1/2}(x_{t+1} - \hat{m}_{t+1})|^2)$$

- Equating quadratic terms in x_{t+1} gives

$$C_{t+1}^{-1} = \hat{C}_{t+1}^{-1} + H^T R^{-1} H$$

- Equating linear terms in x_{t+1} gives

$$C_{t+1}^{-1} m_{t+1} = \hat{C}_{t+1}^{-1} \hat{m}_{t+1} + H^T R^{-1} y_{t+1}$$

Kalman Filter: Derivation of Update Mapping

- Compute C_{t+1} by Sherman-Morrison-Woodbury Formula.

$$C_{t+1} = (C_{t+1}^{-1})^{-1} = (\hat{C}_{t+1}^{-1} + H^T R^{-1} H)^{-1}$$

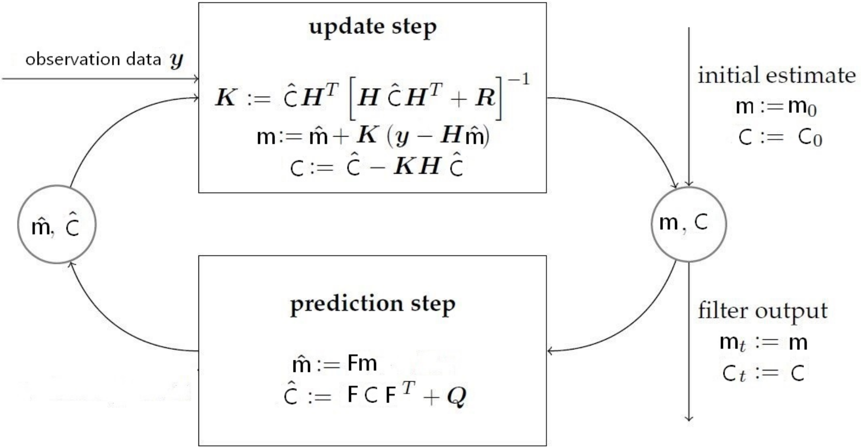
$$\cdots = \hat{C}_{t+1} - (\hat{C}_{t+1} H^T (H \hat{C}_{t+1} H^T + R)^{-1}) H \hat{C}_{t+1}$$

$$= \hat{C}_{t+1} - K_{t+1} H \hat{C}_{t+1}$$

- Compute m_{t+1}

$$m_{t+1} = C_{t+1}(C_{t+1}^{-1} m_{t+1}) = \cdots = \hat{m}_{t+1} + K_{t+1}(y_{t+1} - H\hat{m}_{t+1})$$

Kalman Filter: Iterative Process



Kalman Filter: Properties

- Convergence theorem: The rate of adaptation to new data r is defined by R and Q. As $t \to \infty$, $C_t \to C = AR$ [Mike West and Jeff Harrison,p44]
  where $0 < A < I$
  $$A = \frac{r}{2}\left(\sqrt{1 + \frac{4}{r}} - 1\right)$$

- Complexity:
  - Calculating and storing the $n_x \times n_x$ matrices $\hat{C}_{t+1|t}$ and $\hat{C}_{t+1|t+1}$ are expensive.
  - Calculating the effective of the $n_y \times n_y$ matrix in Kalman Gain is expensive.
  - Computational complexity $O(n_x^3)$

Kalman Filter Approximation: Ensemble Kalman Filter

Generate N ensemble members from initial guess ($N \ll n_x$)

$$\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$$

Propagate in prediction and update steps

$$\{x_t^{(i)}\}_{i=1}^N \xrightarrow{\text{prediction}} \{\hat{x}_{t+1}^{(i)}\}_{i=1}^N \xrightarrow{\text{update}} \{x_{t+1}^{(i)}\}_{i=1}^N$$

The mean m_{t+1} and covariance C_{t+1} of Kalman filter are approximated by ensemble

$$\tilde{m}_{t+1} = \frac{1}{N}\sum_{n=1}^{N} x_{t+1}^{(i)}$$

$$\tilde{C}_{t+1} = \frac{1}{N-1}\sum_{n=1}^{N} (x_{t+1}^{(i)} - \tilde{m}_{t+1})(x_{t+1}^{(i)} - \tilde{m}_{t+1})^T$$

Ensemble Kalman Filter: Properties

Convergence:

- EnKF results converge to the ones in Kalman Filter as $N \to \infty$ under Monte Carlo approximation.
- No proof about convergence in time steps for fixed ensemble size N and unclear about how large of of N is needed for high dimensional approximation.

Complexity:

- Storing and update N vectors with size $n_x \times 1$
- Computational complexity $O(n_x N^2)$
- Applicable in high dimensional state vector.

EnKF: Derivation of Prediction Step

Derive the map $\{x_t^{(i)}\}_{i=1}^N \xrightarrow{\text{prediction}} \{\hat{x}_{t+1}^{(i)}\}_{i=1}^N$

- Given ensemble $\{x_t^{(i)}\}_{i=1}^N$
- Compute predicted ensemble members,

$$\hat{x}_{t+1}^{(i)} = Fx_t^{(i)} + v_t^{(i)}$$

where $v_t^{(i)}$ is a realization of v_t from $N(0, Q)$

As $N \to \infty$, $N(E[\hat{x}_{t+1}^{(i)}], Cov[\hat{x}_{t+1}^{(i)}])$ converges to $N(\hat{m}_{t+1}, \hat{C}_{t+1})$. [Geir Evensen,2003]

EnKF: Derivation of Update Step

Derive the map $\{\hat{x}_{t+1}^{(i)}\}_{i=1}^{N} \xrightarrow{\text{update}} \{x_{t+1}^{(i)}\}_{i=1}^{N}$

- Generate measurement sample using predicted ensemble,

$$\hat{y}_{t+1}^{(i)} = H\hat{x}_{t+1}^{(i)} + w_{t+1}^{(i)}$$

  where $w_{t+1}^{(i)}$ is a realization of w_{t+1} from $N(0, R)$

- Compute ensemble members at present time.

$$x_{t+1}^{(i)} = \hat{x}_{t+1}^{(i)} + K_{t+1}(y_{t+1} - \hat{y}_{t+1}^{(i)})$$

As $N \to \infty$, $N(E[x_{t+1}^{(i)}], Cov[x_{t+1}^{(i)}])$ converges to $N(m_{t+1}, C_{t+1})$. [Geir Evensen,2003]

Ensemble Kalman Filter Algorithm

Initialization: Generate N ensemble members, $\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$

For $t = 1 : T$

1. For $i = 1 : N$
   - $\hat{x}_{t+1}^{(i)} = F(x_t^{(i)}) + v_t^{(i)}$
   - $\hat{m}_{t+1} = \frac{1}{N} \sum_{n=1}^{N} \hat{x}_{t+1}^{(i)}$
   - $\hat{C}_{t+1} = \frac{1}{N-1} \sum_{n=1}^{N} (\hat{x}_{t+1}^{(i)} - \hat{m}_{t+1})(\hat{x}_{t+1}^{(i)} - \hat{m}_{t+1})^T$
2. For $i = 1 : N$
   - $K_{t+1} = \hat{C}_{t+1} H^T (H \hat{C}_{t+1} H^T + R)^{-1}$
   - $\hat{y}_{t+1}^{(i)} = H \hat{x}_{t+1}^{(i)} + w_{t+1}^{(i)}$
   - $x_{t+1}^{(i)} = \hat{x}_{t+1}^{(i)} + K_{t+1}(Y_{t+1} - \hat{y}_{t+1}^{(i)})$

The output at each time step are ensemble members $\{x_{t+1}^{(i)}\}_{i=1}^N$.

The Idea Behind EnKF: Monte Carlo Integration

In estimation of the form (intractable integral)

$$E_p[x_{t+1}|y_{1:t+1}] = \int x_{t+1}|y_{1:t+1} p(x_{t+1}|y_{1:t+1}) dx_{t+1}|y_{1:t+1}$$

$p(x_{t+1}|y_{1:t+1})$ could be approximated by

$$\hat{p}(x_{t+1}|y_{1:t+1}) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_{t+1} - x_{t+1}^{(i)})$$

where $\{x_{t+1}^{(i)}\}_{i=1}^{N}$ are i.i.d. from $p(x_{t+1}|y_{1:t+1})$, In EnKF, $\{x_{t+1}^{(i)}\}_{i=1}^{N}$ are obtained from $\{x_t^{(i)}\}_{i=1}^{N}$ in filtering process.
So the estimation could be approximated by tractable weighted sum:

$$E_p[x_{t+1}|y_{1:t+1}] \approx \frac{1}{N} \sum_{i=1}^{N} x_{t+1}^{(i)}$$

Nonlinear Filtering Problem

Nonlinear dynamic with linear measurement and Gaussian noise:

$$x_{t+1} = f(x_t) + v_t$$

$$y_{t+1} = Hx_{t+1} + w_t$$

- True filtering distribution is non-Gaussian.
- Particle filter algorithms are used in solving nonlinear filtering problems.
  - Pro: Provable of convergence to true filtering distribution.
  - Con: Computationally expensive in high dimensional case.

Monte Carlo Approximation: Importance Sampling

If $p(x)$ is difficult to sample from but easy to evaluate. To have a Monte Carlo approximation of the form,

$$E_p[x] = \int xp(x)dx$$

Choose a proposal density $q(x)$ that is easy to sample from and write

$$E_p[x] = \int xp(x)dx = \int x\frac{p(x)}{q(x)}q(x)dx = E_q[x\omega]$$

Sample from proposal density $\{x^{(i)}\}_{i=1}^N \sim q(x)$ and weight the importance $\omega^{(i)} = p(x^{(i)})/q(x^{(i)})$ The estimation

$$E_p[x] = E_q[x\omega] \approx \sum_{i=1}^{N} x^{(i)}\omega^{(i)}$$

Where $p(x)$ is approximated by $\hat{p}(x) = \sum_{i=1}^N \delta(x - x^{(i)})\omega^{(i)}$

SIR Filter

SIR (Sequential Importance Sampling with Resampling) filter is the simplest particle filter.
Initialization

$$\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$$

The filtering process as follows,

$$\{x_t^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{state prediction}} \{\hat{x}_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{weight update}} \{\hat{x}_{t+1}^{(i)}, \hat{\omega}_{t+1}^{(i)}\}_{i=1}^N$$

$$\xrightarrow{\text{resampling}} \{x_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^N$$

Set proposal density to be $p(x_{t+1}|x_t)$, so that

$$\hat{x}_{t+1}^{(i)} = f(x_t^{(i)}) + v_t^{(i)}$$

SIR Filter: Weight Update Derivation

From Bayes rule,

$$p(x_{0:t+1}|y_{1:t+1}) = p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)p(x_{0:t}|y_{1:t})/p(y_{t+1}|y_{1:t})$$

$$\propto p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)$$

Choose proposal to be transition density function

$$q(x_{0:t+1}|y_{1:t+1}) = p(x_{t+1}|x_t)$$

By importance sampling, the weight is

$$\omega_{t+1} = \frac{p(x_{0:t+1}|y_{1:t+1})}{q(x_{0:t+1}|y_{1:t+1})} \propto p(y_{t+1}|x_{t+1})$$

Given particles $\{\hat{x}_{t+1}^{(i)}\}_{i=1}^{N}$ and data Y_{t+1},

$$\omega_{t+1}^{(i)} \propto p(Y_{t+1}|\hat{x}_{t+1}^{(i)})$$

SIR Filter: Degeneracy

- Particle Degeneracy: the variance of weights increase over time [Kong and Liu, 1994]
  Degeneracy can be measured by

$$N_{ess} = \frac{N}{1 + Var(\omega_{t+1}^{(i),true})} \text{ Or } \hat{N}_{ess} = \frac{1}{\sum_{i=1}^{N}(\omega_{t+1}^{(i)})^2}$$

- Resampling to reduce the effect of degeneracy.

$$\{\hat{x}_{t+1}^{(i)}, \hat{\omega}_{t+1}^{(i)}\}_{i=1}^{N} \xrightarrow{\text{resampling}} \{x_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^{N}$$

- Monte Carlo methods require effective samples $N_{ess} \to \infty$ to ensure the convergence to true distribution.
- Computational power is wasted on particles with zero weight.

SIR Filter: Resampling

1. Goal of resampling

$$\{\hat{x}_{t+1}^{(i)}, \hat{\omega}_{t+1}^{(i)}\}_{i=1}^{N} \xrightarrow{\text{resampling}} \{x_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^{N}$$

2. Define

$$A = \sum_{i=1}^{N} \hat{\omega}_{t+1}^{(i)}$$

3. Generate N random numbers $\theta_k, k = 1, \ldots, N$ from uniform distribution $U(0, 1)$
4. Choose

$$x_{t+1}^{(k)} = \hat{x}_{t+1}^{(i)}$$

such that

$$A^{-1} \sum_{j=1}^{i-1} \hat{\omega}_{t+1}^{(j)} < \theta_k \leq A^{-1} \sum_{j=1}^{i} \hat{\omega}_{t+1}^{(j)}$$

SIR Filter: Algorithm

Initialize particle $\{x_0^{(i)}\}_{i=1}^{N} \sim N(m_0, C_0)$
For $t = 0 : T$

1. For each particle $i = 1 : N$
   - State prediction:
   $$\hat{x}_{t+1}^{(i)} = f(x_t^{(i)}) + v_t^{(i)}$$
   where $v_t^{(i)} \sim N(0, Q)$
   - Weight update:
   $$\omega_{t+1}^{(i)} \propto exp(-\frac{1}{2}|R^{-1/2}(Y_{t+1} - H(\hat{x}_{t+1}^{(i)}))|^2)$$
2. Resample.

The Choice of Proposal Density Function

- Good choice of proposal density also reduce the effect of degeneracy.
- Introduce an auxiliary variable j which denote the j-th particle of time t.
- By Bayes theorem,

$$p(x_{t+1}, j | y_{1:t+1}) \propto p(y_{t+1} | x_{t+1}) p(x_{t+1} | x_t^{(j)})$$

- Define the proposal density close to the joint filtering density,

$$q(x_{t+1}, j | y_{1:t+1}) \propto p(y_{t+1} | \mu_{t+1}^{(j)}) p(x_{t+1} | x_t^{(j)})$$

where $\mu_{t+1}^{(j)} = E[x_{t+1} | x_t^{(j)}]$

Auxiliary Particle Filter

- Initialization $\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$
- Process to sample the auxiliary variable use present data Y_{t+1}

$$\{x_t^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{look forward}} \{\mu_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{weight update}} \{\mu_{t+1}^{(i)}, \tilde{\omega}_t^{(i)}\}_{i=1}^N$$

$$\{x_t^{(i)}, \tilde{\omega}_t^{(i)}\}_{i=1}^N \xrightarrow{\text{resampling}} \{x_t^{(j)}, \frac{1}{N}\}_{j=1}^N$$

- Filtering process

$$\{x_t^{(j)}, \frac{1}{N}\}_{j=1}^N \xrightarrow{\text{prediction}} \{\hat{x}_{t+1}^{(j)}, \frac{1}{N}\}_{j=1}^N$$

$$\xrightarrow{\text{update}} \{\hat{x}_{t+1}^{(j)}, \omega_{t+1}^{(j)}\}_{j=1}^N \xrightarrow{\text{resampling}} \{x_{t+1}^{(j)}, \frac{1}{N}\}_{j=1}^N$$

APF: Derivation of Weight of Auxiliary Variable

- Look forward of state and use present data Y_{t+1} to update the weight.

$$\{x_t^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{look forward}} \{\mu_{t+1}^{(i)}, \frac{1}{N}\}_{i=1}^N \xrightarrow{\text{weight update}} \{\mu_{t+1}^{(i)}, \tilde{\omega}_t^{(i)}\}_{i=1}^N$$

- Use dynamic equation to make a prediction

$$\mu_{t+1}^{(i)} = f(x_t^{(i)})$$

- The weight $\tilde{\omega}_t^{(i)}$ is obtained by combining the following two equations.

$$\mathit{Definition} : q(x_{t+1}, i | Y_{1:t+1}) \propto p(Y_{t+1} | \mu_{t+1}^{(i)}) p(x_{t+1} | x_t^{(i)})$$

$$\begin{aligned} \mathit{Conditional} : q(x_{t+1}, i | Y_{1:t+1}) &= q(i | Y_{1:t+1}) q(x_{t+1} | i, Y_{1:t+1}) \\ &= q(i | Y_{1:t+1}) p(x_{t+1} | x_t^{(i)}) \end{aligned}$$

- Therefore,

$$\tilde{\omega}_t^{(i)} = q(i | Y_{1:t+1}) \propto p(Y_{t+1} | \mu_{t+1}^{(i)})$$

APF: filtering process

Filtering process:

$$\{x_t^{(i)}, \tilde{\omega}_t^{(i)}\}_{i=1}^N \xrightarrow{\text{resampling}} \{x_t^{(j)}, \frac{1}{N}\}_{j=1}^N \xrightarrow{\text{prediction}} \{\hat{x}_{t+1}^{(j)}, \frac{1}{N}\}_{j=1}^N$$

$$\xrightarrow{\text{update}} \{\hat{x}_{t+1}^{(j)}, \omega_{t+1}^{(j)}\}_{j=1}^N \xrightarrow{\text{resampling}} \{x_{t+1}^{(j)}, \frac{1}{N}\}_{j=1}^N$$

1. Resample to obtain particle $\{x_t^{(j)}\}_{j=1}^N$
2. Propagate in dynamic equation:

$$\hat{x}_{t+1}^{(j)} = f(x_t^{(j)}) + v_t^{(j)}$$

3. Weight update

$$\omega_{t+1}^{(j)} \propto \frac{p(Y_{t+1}|\hat{x}_{t+1}^{(j)})}{p(Y_{t+1}|\mu_{t+1}^{(j)})}$$

4. Resampling to eliminate the variance of weight.

Auxiliary Particle Filter (ASIR) Algorithm

Initialization $\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$
For $t = 0 : T$

1. Given $\{x_t^{(i)}\}_{i=1}^N$, For $i = 1 : N$
   - State Prediction:$\mu_{t+1}^{(i)} = f(x_t^{(i)})$
   - Weight Update: $\tilde{\omega}_t^{(i)} = N(Y_{t+1}|H\mu_{t+1}^{(i)}, R)$
2. Resampling for $\{x_t^{(i)}, \tilde{\omega}_t^{(i)}\}_{i=1}^N$ to obtain $\{x_t^{(j)}, 1/N\}_{j=1}^N$
3. For each particle, given $x_t^{(j)}$
   - state prediction:
   $$\hat{x}_{t+1}^{(j)} = f(x_t^{(j)}) + v_t^{(j)}$$
   where $v_t^{(j)} \sim N(0, Q)$
   - weight update: $\omega_{t+1}^{(j)} = N(Y_{t+1}|H\hat{x}_{t+1}^{(j)}, R)/N(Y_{t+1}|H\mu_{t+1}^{(j)}, R)$
4. Resample $\{\hat{x}_{t+1}^{(j)}, \omega_{t+1}^{(j)}\}_{j=1}^N$ to obtain $\{x_{t+1}^{(j)}, 1/N\}_{j=1}^N$

Implicit Particle Filter (IPF)

- From the Bayes theorem,

$$p(x_{0:t+1}|y_{1:t+1}) = p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)p(x_{0:t}|y_{1:t})/p(y_{t+1}|y_{1:t})$$

- Given particle $\{x_{0:t}^{(i)}, \frac{1}{N}\}_{i=1}^{N}$, and data $Y_{1:t+1}$, we have a expression of x_{t+1}

$$p(x_{0:t}^{(i)}, x_{t+1}|Y_{1:t+1}) \propto p(Y_{t+1}|x_{t+1})p(x_{t+1}|x_t^{(i)})$$

- Objective: Sample $x_{t+1}^{(i)}$ on high probability region of the posterior $p(x_{0:t}^{(i)}, x_{t+1}|Y_{1:t+1})$.

IPF: Obtain High Probability Particles

- General Idea: Rather than find particles from better proposal density and then estimate their probability, first pick a probability and then find a sample that carries it.
- Given $\{x_t^{(i)}\}_{i=1}^N$, sample $\{x_{t+1}^{(i)}\}_{i=1}^N$ as follows:
  1. Pick sample $\xi_{t+1}^{(i)}$ from a known, fixed, pdf. e.g. a Gaussian $N(0, I)$, $p(\xi) = exp(-\frac{1}{2}\xi_{t+1}^T\xi_{t+1})/(2\pi)^{n_x/2}$
  2. Write the posterior as $p(Y_{t+1}|\mathbf{x}_{t+1}^{(i)})p(\mathbf{x}_{t+1}^{(i)}|x_t^{(i)})$ in the form $exp(-F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}))$
  3. To obtain high probability particles, solve

$$F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) - \min F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) = \frac{1}{2}((\xi_{t+1}^{(i)})^T\xi_{t+1}^{(i)})$$

- The right pdf is sampled if map $\xi_{t+1}^{(i)} \to \mathbf{x}_{t+1}^{(i)}$ is one-to-one and onto.

IPF: Quadratic Approximation of $F_{t+1}^{(i)}$

With nonlinear dynamics and linear observation in the model

- Quadratic approximation of $F_{t+1}^{(i)}$ is a formula of $\mathbf{x}_{t+1}^{(i)}$

$$F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) = \frac{1}{2}(\mathbf{x}_{t+1}^{(i)} - f(x_t^{(i)}))^T Q^{-1}(\mathbf{x}_{t+1}^{(i)} - f(x_t^{(i)}))$$

$$+\frac{1}{2}(y_{t+1} - H\mathbf{x}_{t+1}^{(i)})^T R^{-1}(y_{t+1} - H\mathbf{x}_{t+1}^{(i)})$$

- Completing the square,

$$F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) = \frac{1}{2}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)})^T \Sigma^{-1}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)}) + \phi_{t+1}^{(i)}$$

where

$\Sigma^{-1} = (Q^T Q)^{-1} + H^T(R^T R)H$

$\bar{m}_{t+1}^{(i)} = \Sigma((Q^T Q)^{-1}(x_t^{(i)}) + H(Q^T Q)^{-1} Y_{t+1})$

$\phi_{t+1}^{(i)} = \frac{1}{2}(Y_{t+1} - Hf(x_{t+1}^{(i)}))^T(HQ^T QH^T + R^T R)^{-1}(Y_{t+1} - Hf(x_{t+1}^{(i)}))$

IPF: Simplification of Equation

- From

$$F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) = \frac{1}{2}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)})^T \Sigma^{-1} (\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)}) + \phi_{t+1}^{(i)}$$

- It follows that

$$\min F_{t+1}^{(i)}(\mathbf{x}_{t+1}^{(i)}) = \phi_{t+1}^{(i)}$$

- Therefore

$$\frac{1}{2}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)})^T \Sigma^{-1} (\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)}) = \frac{1}{2}\xi_{t+1}^{(i)T} \xi_{t+1}^{(i)}$$

IPF: Solving the Underdetermined Equation

- One equation n_x unknowns

$$\frac{1}{2}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)})^T \Sigma^{-1} (\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)}) = \frac{1}{2} \xi_{t+1}^{(i)T} \xi_{t+1}^{(i)}$$

- A one-to-one mapping $\xi \to x$ is

$$L^{-1}(\mathbf{x}_{t+1}^{(i)} - \bar{m}_{t+1}^{(i)}) = \xi_{t+1}^{(i)} \text{ Or } \mathbf{x}_{t+1}^{(i)} = \bar{m}_{t+1}^{(i)} + L\xi_{t+1}^{(i)}$$

where $\Sigma = LL^T$ is Cholesky decomposition.

IPF: Find the Proposal Density

- The probability density of reference variable ξ is Gaussian by our choice.
- The proposal density is,

$$q(x_{t+1}^{(i)}) = \frac{p(\xi_{t+1}^{(i)})}{J} \propto \frac{exp(-\frac{1}{2}\xi_{t+1}^{(i)\,T}\xi_{t+1}^{(i)})}{J}$$

$$= \frac{exp(\phi_{t+1}^{(i)} - F(x_{t+1}^{(i)}))}{J} = \frac{exp(\phi_{t+1}^{(i)})}{J} p(Y_{t+1}|x_{t+1}^{(i)})p(x_{t+1}^{(i)}|x_t^{(i)})$$

where J is the determinant of Jacobian matrix.
- From the mapping

$$x_{t+1}^{(i)} = \bar{m}_{t+1}^{(i)} + L\xi_{t+1}^{(i)}$$

We have

$$J = |\det(\frac{\partial x_{t+1}^{(i)}}{\partial \xi^{(i)}})| = |\det L|$$

IPF: Find the Particle Weights

By importance sampling,

$$\omega_{t+1}^{(i)} = \frac{p(x_{0:t+1}^{(i)}|Y_{1:t+1})}{q(x_{t+1}^{(i)})}$$

$$\propto \frac{p(Y_{t+1}|x_{t+1}^{(i)})p(x_{t+1}^{(i)}|x_t^{(i)})}{exp(\phi_{t+1}^{(i)})p(Y_{t+1}|x_{t+1}^{(i)})p(x_{t+1}^{(i)}|x_t^{(i)})/|\det L|}$$

$$\propto exp(-\phi_{t+1}^{(i)})|\det L|$$

IPF: Algorithm

Initialization $\{x_0^{(i)}\}_{i=1}^N \sim N(m_0, C_0)$
Compute L by Cholesky decomposition $\Sigma = LL^T$
For $t = 0 : T$

1. Given $\{x_t^{(i)}\}_{i=1}^N$, For $i = 1 : N$
   - Generate $\xi^{(i)} \sim N(0, I)$
   - Compute $\bar{m}^{(i)}, \phi_{t+1}, |\det L|$
   - State Prediction:$\hat{x}_{t+1}^{(i)} = \bar{m}^{(i)} + L\xi^{(i)}$
   - Weight Update: $\omega_{t+1}^{(i)} = exp(-\phi_{t+1}^{(i)})|\det L|$
2. Resampling for $\{\hat{x}_{t+1}^{(i)}, \omega_{t+1}^{(i)}\}_{i=1}^N$ to obtain $\{x_{t+1}^{(i)}, 1/N\}_{i=1}^N$

Experiment: Objective

Compare the performance of filtering algorithms with same sample size,

- The number of times steps for the estimation to track 'on target' from initial guess.
- The accuracy of estimation at certain time step, e.g. $t = 100$.

Experiment

One dimensional particle moving in the potential

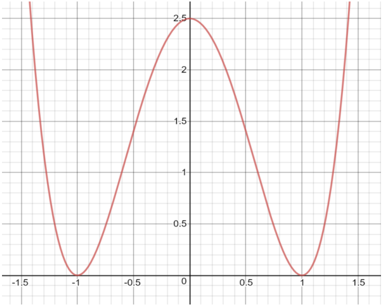
$$V(x) = \alpha(x^2 - 1)^2$$

With the force

$$-\nabla V(x) = 4\alpha(x - x^3)$$

The resulting SDE

$$\frac{dx}{dt} = 4\alpha(x - x^3) + u, u \sim N(0, q)$$



Experiment

- Discretization in time by Euler scheme.

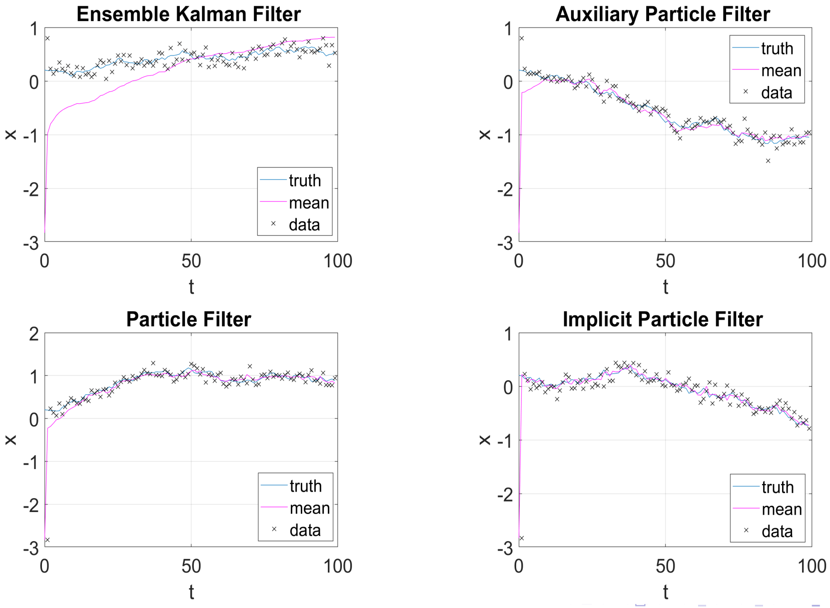
$$x_{t+1} = x_t + 4\alpha(x_t - x_t^3)\delta_t + v_t, v_t \sim N(0, q\delta_t)$$

- Define the linear measurement equation

$$y_{t+1} = x_{t+1} + w, w \sim N(0, r)$$

- Set initial state $x_0 \sim N(0, \sigma_0)$
- $\alpha = 2.5, \delta_t = 0.02, q = 0.3, r = 0.1, \sigma_0 = 10$

Experiment: Convergence in Time



Accuracy of Estimation

- Compare the RMSE at $t = 100$ by simulating the process 1000 times with particle size $N = 20$

$$RMSE = \sqrt{\frac{1}{1000}\sum_{j=1}^{1000}(x_{100}^{true,j} - m_{100}^{j})^2} \text{ ,where } m_{100} = \sum_{i=1}^{20} x_{100}^{(i)}$$

- Average effective sample size is

$$\text{Average } N_{ess} = \frac{1}{1000}\sum_{j=1}^{1000} N_{ess}^{j} \text{ ,where } N_{ess} = \frac{1}{\sum_{i=1}^{20}(\omega_{t+1}^{(i)})^2}$$

	RMSE	**average effective sample size**
EnKF	0.093	20
SIR	0.064	14.73
APF	0.051	19.19
IPF	0.048	18.33

Conclusion and Future Work

- Conclusion: reviewed five filtering algorithms for state estimation.

- Future Work:
  - Apply the idea of implicit sampling to more sophisticated problems (high dimensional, nonlinear observations, non-Gaussian noise)
  - Derive filtering methods for both parameter and state estimation.

Reference

[1] Chorin A. and Tu X., Implicit Sampling for Particle Filters, P.N.A.S. 106: 17249-17254 (2009)
[2] Chorin A., Morzfeld M., and Tu X., Implicit Particle Filters for Data Assimilation, Commun. Appl. Math. Comput. Sci. 5: 221-240 (2010)
[3] Chorin A. and Tu X., Interpolation and Iteration for Nonlinear Filters, M2AN Math. Model. Numer. Anal. 46: 535-543 (2012)
[4] Augustine Kong; Jun S. Liu; Wing Hung Wong, Sequential Imputations and Bayesian Missing Data Problems, Journal of the American Statistical Association, Vol. 89, No. 425. (Mar., 1994), pp. 278-288.
[5] Arnaud Doucet,Nando de Freitas,Neil Gordon, Sequential Monte Carlo Methods in Practice, Springer-Verlag New York 2001
[6] Arnaud Doucet , Adam M. Johansen, A tutorial on particle filtering and smoothing: fifteen years later (2011)
[7] Adam M. Johansen, Arnaud Doucet, A Note on Auxiliary Particle Filters
[8] Geir Evensen, The Ensemble Kalman Filter: theoretical formulation and practical implementation, Ocean Dynamics (2003) 53: 343367